8

DISTRIBUTED CHARGE SYSTEMS

Introduction. We have recently been studying *solutions* of Maxwell's equations —solutions in the complete absence of sources (Chapter 5) and solutions in the presence of but a single point source (Chapters 6 & 7). But in many physical problems and most technological applications one has interest in the fields generated by (static or dynamic) *populations* of charged particles; *i.e.*, by spatially *distributed sources*.

One might suppose that such problems could be solved by application of the principle of superposition . . . but the "application" is more easily talked about than done, and it is not at all straightforward: it inspired much of the mathematical invention for which the period 1775–1875 is remembered. And there are (as always) unexpected physical complications. For example: the presence of conductive materials gives rise to "*induced* charges," which join the unknowns of the problem.

We will look first to the electro*static* problem—to the description of the description of the electrostatic potential set up by an arbitrarily constructed blob of charge. Information of the sort we now seek would comprise our point of departure if se sought (say) to construct an account of the Bohr orbits around a structured nucleus, or (in gravitational terms) of the motion of a satellite around the inhomogenous earth.

1. Multipole representation of a static source. Let $\rho(\boldsymbol{x})$ describe a t-independent (or "static") charge distribution. The resulting electromagnetic field has no magnetic component ($\boldsymbol{B} = \boldsymbol{0}$), and its t-independent electric component (see again page 25) can be described

$$\begin{aligned} \boldsymbol{E}(\boldsymbol{x}) &= -\boldsymbol{\nabla}\varphi(\boldsymbol{x}) \\ \varphi(\boldsymbol{x}) &= \tfrac{1}{4\pi}\iiint \rho(\boldsymbol{x})\,\frac{1}{|\boldsymbol{x}-\boldsymbol{x}|}\,d^3x \end{aligned} \tag{506}$$

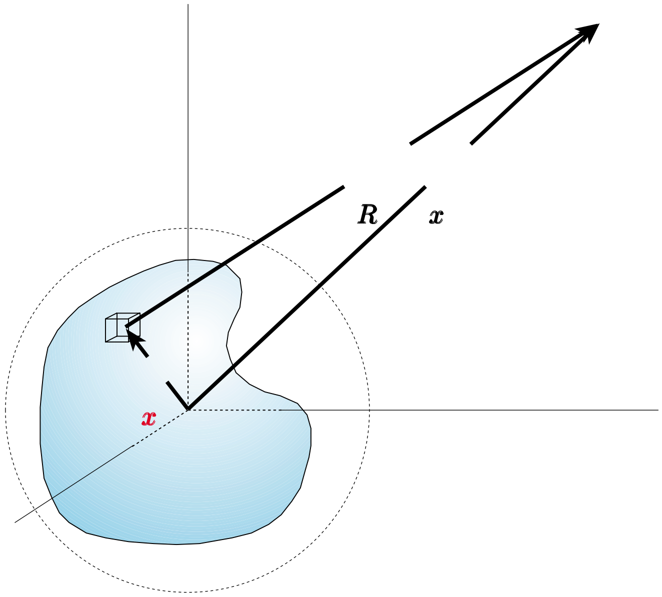


FIGURE 153: *We use* $\boldsymbol{x}$ *to describe the constituent elements of a distributed charge, and* $\boldsymbol{x}$ *to describe the location of a typical field point. The vector* $\boldsymbol{R}(\boldsymbol{x},\boldsymbol{x}) \equiv \boldsymbol{x} - \boldsymbol{x}$ *stretches from the former to the latter, and has length* $R(\boldsymbol{x},\boldsymbol{x}) = |\boldsymbol{x} - \boldsymbol{x}|$. *We proceed in the assumption that* $r \equiv \sqrt{\boldsymbol{x}\cdot\boldsymbol{x}} > a$, *where*

$$a \equiv \begin{cases} \text{radius of a mental sphere large enough} \\ \text{to enclose the entire distributed charge} \end{cases}$$

The integral $\iiint$ derives from, and expresses, the principle of superposition—as anticipated. But our goal now is to see what we can do to *sharpen* the very general result described above. We want to learn to distinquish the relevant features of (506) from the less relevant, so that by discarding the latter we can *simplify our computational life*.

Let us suppose that the source, though distributed, is "localized" in the sense that $\rho(\boldsymbol{x}) \equiv 0$ for $\boldsymbol{x}$ exterior to a sphere of sufficiently large but finite radius a,[309] and let us agree that our ultimate objective—what we are presently getting in position to do—is to describe the electrostatic potential at points *external* to that sphere (see Figure 153). Writing

$$\begin{aligned} R(\boldsymbol{x},\boldsymbol{x}) = |\boldsymbol{x} - \boldsymbol{x}| &= \sqrt{(\boldsymbol{x} - \boldsymbol{x})\cdot(\boldsymbol{x} - \boldsymbol{x})} \\ &= \sqrt{r^2 - 2rr\cos\vartheta + r^2} \end{aligned}$$

[309] This weak assumption serves merely to exclude "infinite line charges" and similar (unphysical) abstractions.

with $r \equiv \sqrt{\boldsymbol{x}\cdot\boldsymbol{x}}$ and $r \equiv \sqrt{\boldsymbol{x}\cdot\boldsymbol{x}}$, we note that the dimensionless ratios x/r, y/r, z/r are in every instance less than unity. It becomes therefore natural to contemplate expanding $1/R(\boldsymbol{x},\boldsymbol{x})$ in powers of those ratios. To that end ... we recall that according to Taylor's theorem

$$f(x+x) = e^{x\frac{\partial}{\partial x}}f(x) = \sum_{n=0}^{\infty}\tfrac{1}{n!}x^n f^{(n)}(x)$$

In the multivariate case we expect therefore to have

$$\begin{aligned}
f(x+x,y+y,z+z) &= e^{x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}}f(x,y,z)\\
&= \Big\{1+\big[x\tfrac{\partial}{\partial x}+y\tfrac{\partial}{\partial y}+z\tfrac{\partial}{\partial z}\big]\\
&\qquad +\tfrac{1}{2}\big[x^2\tfrac{\partial^2}{\partial x^2}+2xy\tfrac{\partial^2}{\partial x\partial y}+2xz\tfrac{\partial^2}{\partial x\partial z}\\
&\qquad\quad +y^2\tfrac{\partial^2}{\partial y^2}+2yz\tfrac{\partial^2}{\partial y\partial z}+z^2\tfrac{\partial^2}{\partial z^2}\big]+\cdots\Big\}f(x,y,z)
\end{aligned}$$

which when applied in particular to the $\boldsymbol{x}$-dependence of $1/R(\boldsymbol{x},\boldsymbol{x})$ gives

$$\begin{aligned}
\frac{1}{|\boldsymbol{x}-\boldsymbol{x}|} &= \frac{1}{r}+\frac{1}{r^3}\cdot\big[xx+yy+zz\big]\\
&\quad +\frac{1}{r^5}\cdot\tfrac{1}{2}\big[x^2(3x^2-r^2)+6xyxy+6xzxz\\
&\qquad +y^2(3y^2-r^2)+6yzyz+z^2(3z^2-r^2)\big]+\cdots
\end{aligned}$$

In a fairly natural (and quite useful) condensed notation we have

$$\begin{aligned}
&= r^{-1}+r^{-3}\begin{pmatrix}x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix}\\
&\quad + r^{-5}\tfrac{1}{2}\begin{pmatrix}x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}3x^2-r^2 & 3xy & 3xz\\ 3yx & 3y^2-r^2 & 3yz\\ 3zx & 3zy & 3z^2-r^2\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}+\cdots
\end{aligned}$$

Feeding this expansion back into (506) we obtain

$$\begin{aligned}
\varphi(\boldsymbol{x}) &= \tfrac{1}{4\pi}\Big\{r^{-1}q+r^{-3}\boldsymbol{p}\cdot\boldsymbol{x}+r^{-5}\tfrac{1}{2}\boldsymbol{x}\cdot\mathbb{Q}\boldsymbol{x}+\cdots\Big\}\\
&= \tfrac{1}{4\pi}\Big\{r^{-1}q+r^{-2}\boldsymbol{p}\cdot\hat{\boldsymbol{x}}+r^{-3}\tfrac{1}{2}\hat{\boldsymbol{x}}\cdot\mathbb{Q}\hat{\boldsymbol{x}}+\cdots\Big\}
\end{aligned} \tag{508}$$

where

$$q \equiv \iiint \rho(\boldsymbol{x})\, d^3x \tag{508.0}$$

$\equiv$ so-called "monopole moment scalar" or **total charge**

$$\boldsymbol{p} \equiv \iiint \begin{pmatrix} x \\ y \\ z \end{pmatrix} \rho(\boldsymbol{x})\, d^3x \tag{508.1}$$

$\equiv$ so-called "**dipole moment vector**"

$$\mathbb{Q} \equiv \iiint \begin{pmatrix} 3x^2 - r^2 & 3xy & 3xz \\ 3yx & 3y^2 - r^2 & 3yz \\ 3zx & 3zy & 3z^2 - r^2 \end{pmatrix} \rho(\boldsymbol{x})\, d^3x \tag{508.2}$$

$\equiv$ so-called "**quadrupole moment** (tensor or) **matrix**"

In higher order we lose the advantages of matrix notation . . . might appear in 3rd order to have to write something like

$$\tfrac{1}{4\pi} r^{-4} \tfrac{1}{3!} \sum_{\substack{a,b,c \\ a+b+c=3}} W_{abc}\, \hat{x}^a \hat{x}^b \hat{x}^c \quad \text{with} \quad W_{abc} \equiv \iiint \underbrace{M_{abc}(x,y,z)}_{\text{complicated cubic}} \rho(\boldsymbol{x})\, d^3x$$

but will soon be in position to proceed in a more orderly manner. As will emerge, it is the lowest-order terms that are of highest practical importance, so (508) is in fact quite useful as it stands: it will be useful also as a benchmark against which to test more general formulæ as they become available. Several comments are now in order:

1. The objects q, $\boldsymbol{p}$, $\mathbb{Q}$, . . . are called "scalar," "vector," "tensor,". . . in recognition of how they respond to rotations of the Cartesian frame: they are, in short, *tensorial with respect to the rotation group* $O(3)$, as one could demonstrate without difficulty.

2. q is the 0th moment of the charge distribution $\rho(\boldsymbol{x})$, $\boldsymbol{p}$ is assembled from the 1st moments, $\mathbb{Q}$ is assembled from the 2nd moments, *etc*. Not surprisingly, *if one possessed the moments of all orders then one could reconstruct the* $\rho(\boldsymbol{x})$ *which generated those moments*.[310]

3. $\mathbb{Q}$ is (like the energy/momentum tensor $\mathbb{S}$: see again page 215) *symmetric* and *traceless*. These properties are, moreover, preserved under coordinate

---

[310] Usually, not always. The program would fail if, for example (see again page 416), the distribution were Lorentzian

$$\rho(\boldsymbol{x}) \sim \frac{1}{x^2 + y^2 + z^2 + a^2}$$

But such a distribution cannot be enclosed within a sphere of finite radius.

rotation. From symmetric tracelessness it follows that $\mathbb{Q}$ contains (not 9, as one would otherwise expect, but) only 5 adjustable constants (degrees of freedom). Symmetry alone assures that $\mathbb{Q}$ can always be rotated to diagonal form

$$\mathbb{Q} \xrightarrow[\text{properly chosen rotation}]{} \begin{pmatrix} Q_1 & 0 & 0 \\ 0 & Q_2 & 0 \\ 0 & 0 & Q_3 \end{pmatrix}$$

and tracelessness requires that the eigenvalues sum to zero: $Q_1 + Q_2 + Q_3 = 0$.

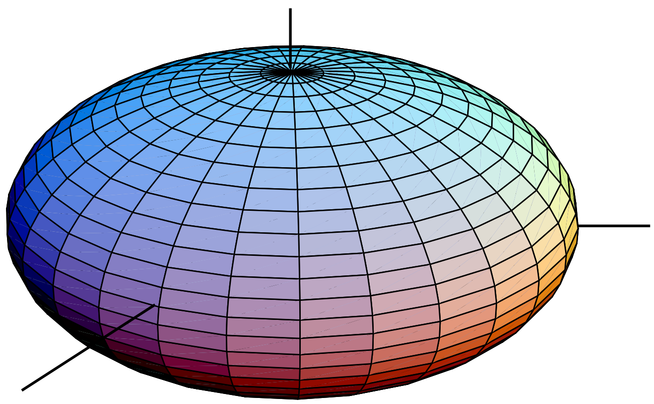


FIGURE 154: *Oblate spheroidal distribution, symmetric about the z-axis. Spinning bodies (stars, planets, atomic nuclei) commonly possess this shape, at least in leading approximation.*

If, as is quite commonly the case, $\rho(\boldsymbol{x})$ is symmetric about the z-axis (see the figure) then $\mathbb{Q}$ acquires the structure

$$\begin{pmatrix} -\frac{1}{2}Q & 0 & 0 \\ 0 & -\frac{1}{2}Q & 0 \\ 0 & 0 & Q \end{pmatrix}$$

In such specialized contexts it is common (among nuclear physicists and others) to speak of "*the* quadrupole moment," the reference being to Q.

4. What is the origin of the monopole/dipole/... **multipole** terminology? The answer has little/nothing to do with electrostatics *per se*, much to do with the meaning of n^{th} derivative. Look, for example, a 1-dimensional model of the situation in hand: suppose it to be the case that

$$\varphi(x) = \int \rho(x)F(x - x)\,dx$$

where $F(\bullet)$ is some prescribed differentiable function (*not* necessarily the $\frac{1}{x-x}$ encountered in (506)) and where x remains "small" throughout the range of integration. We expect then to have

$$\varphi(x) = \sum_{n=0}^{\infty}(-)^n \tfrac{1}{n!}\cdot \underbrace{\int \rho(x)x^n dx}_{n^{\text{th}}\text{ moment}} \cdot F^{(n)}(x)$$

where $F^{(0)}(x)$, $F^{(1)}(x)$, $F^{(2)}(x)$, $F^{(3)}(x)$, . . . acquire meaning from the following scheme:

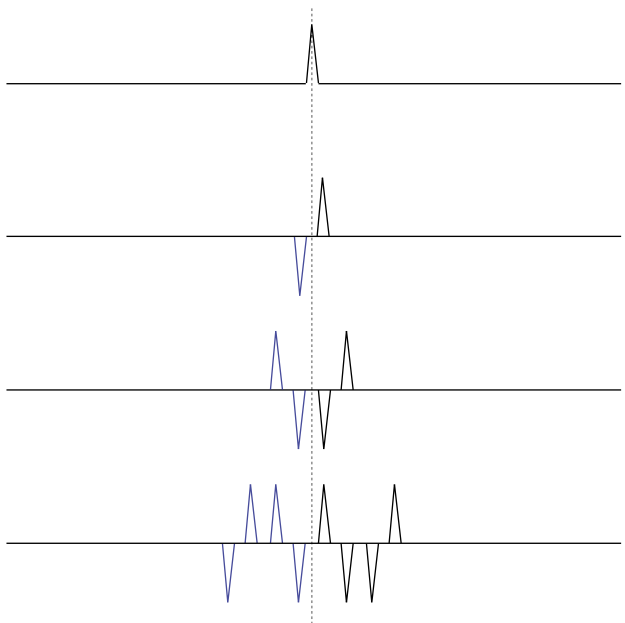


FIGURE 155: *Representation of the mechanism by which iteration of*

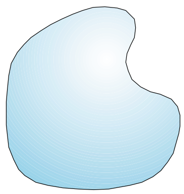
$$F^{(1)}(x) = \lim_{\epsilon\downarrow 0}\int \frac{\delta\big(\xi - (x+\frac{1}{2}\epsilon)\big) - \delta\big(x - (x-\frac{1}{2}\epsilon)\big)}{\epsilon}F(\xi)\,d\xi$$

gives rise to successive derivatives of $F(x)$*. Notice that* 2^n *spikes contribute to the construction of* $F^{(n)}(x)$*. This is the source of the "di/quadu/octo. . .* 2^n*-tuple pole" terminology.*

In several dimensions one encounters only this new circumstance: one can displace a sign-reversed monopole in several directions to create a dipole, can displace a sign-reversed dipole in several directions to create a quadrupole, *etc.*

5. We are led thus to the principle that an arbitrary localized distribution $\rho(\boldsymbol{x})$ can be represented as the superposition of

- an appropriately selected monopole +
- an appropriately selected dipole +
- an appropriately selected quadrupole + *etc*:

 $= \text{monopole} + \text{dipole} + \text{quadrupole} + \cdots$

6. Looking back again to (508) we notice that at sufficiently remote field points one can drop all but the monopole term ($\rho(\boldsymbol{x})$ looks like a point charge). At less remote points one can drop all terms subsequent to the dipole term. *High order multipole terms depend upon such high powers of* $1/r$ *that they are of quantitative importance only in the near zone*.

Equation (508) carries us a long way toward our goal, as stated on page 422. But there remains a good deal of meat to be gnawed from the bone.

2. Electrostatic potential of a dipole. Consider the two-charge configuration (no *net* charge) shown in Figure 156. The associated electrostatic potential can be described

$$\begin{aligned}\varphi(\boldsymbol{x}) &= \tfrac{1}{4\pi}q\Big\{\frac{1}{\sqrt{r^2-2ra\cos\vartheta+a^2}}-\frac{1}{\sqrt{r^2+2ra\cos\vartheta+a^2}}\Big\} &(509.1)\\ &= \tfrac{1}{4\pi}(q/r)\Big\{\Big[1-2\tfrac{a}{r}\cos\vartheta+\big(\tfrac{a}{r}\big)^2\Big]^{-\frac{1}{2}}-\Big[1+2\tfrac{a}{r}\cos\vartheta+\big(\tfrac{a}{r}\big)^2\Big]^{-\frac{1}{2}}\Big\}\\ &= \tfrac{1}{4\pi}\frac{2qa\cos\vartheta}{r^2}\Big\{1+\frac{5\cos2\vartheta-1}{4}\big(\tfrac{a}{r}\big)^2 &(509.2)\\ &\qquad\qquad+\frac{63\cos4\vartheta-28\cos2\vartheta+29}{64}\big(\tfrac{a}{r}\big)^4+\cdots\Big\}\end{aligned}$$

This describes, as a power series in a/r, the potential of a *physical* dipole. Proceeding now to the double limit

$$a\downarrow 0 \text{ and } q\uparrow\infty \text{ in such a way that } p\equiv 2aq \text{ remains constant}$$

we obtain

$$\begin{aligned}&\downarrow\\ &= \tfrac{1}{4\pi}\frac{p\cos\vartheta}{r^2} = \tfrac{1}{4\pi}\frac{\boldsymbol{p}\cdot\hat{\boldsymbol{x}}}{r^2} = \tfrac{1}{4\pi}\frac{\boldsymbol{p}\cdot\boldsymbol{x}}{r^3}\end{aligned} \qquad (510)$$

Notice that the dipole potential φ would simply vanish if q were held *constant* during the compression process $a\downarrow 0$. Equipotentials derived from (509) and (510) are shown in Figure 157.

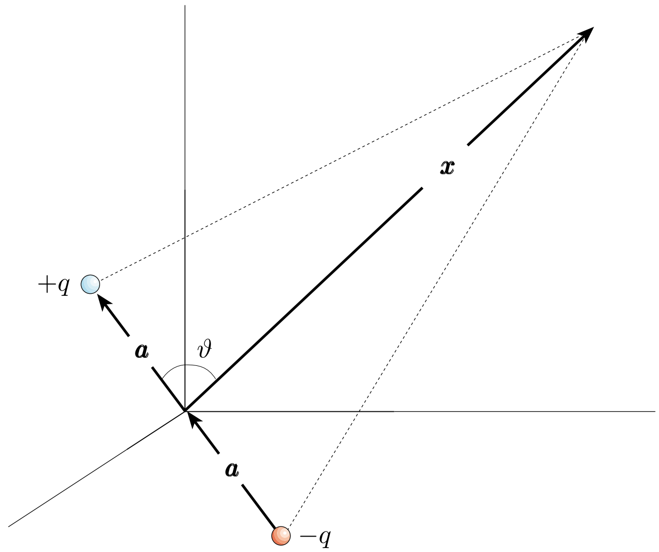


FIGURE 156: *Notation used in the text to describe the field of a physical dipole* •—•. *A "mathematical dipole" results in the idealized limit* $a \downarrow 0$, $q \uparrow \infty$ *with* $p \equiv 2aq$ *held constant.*

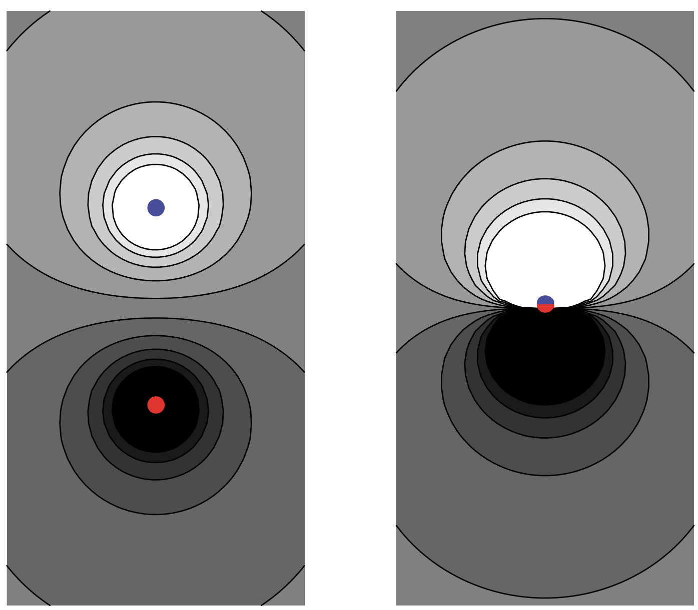


FIGURE 157: *Central cross section of the equipotentials of a physical dipole (on the left) and of an idealized dipole (on the right).*

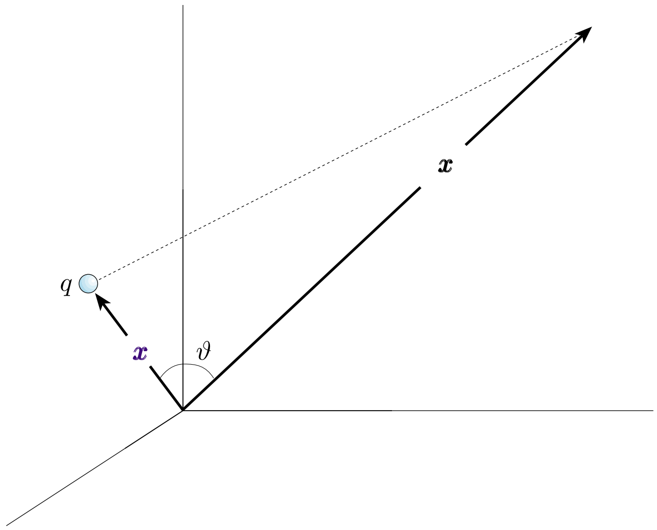


FIGURE 158: *Notation used in the text to describe the field of an "eccentric monopole," i.e., of an isolated charge (or charge element) that is arbitrarily positioned with respect to the coordinate origin. The length of* $\boldsymbol{x}$ *is* r*, the length of* $\boldsymbol{x}$ *is* r.

3. Electrostatic potential of an eccentric monopole. In what might at first sight appear to be a step backward, but will soon be revealed to be a long step forward, we look now to the potential of the primitive system shown above; *i.e.*, to the Coulomb potential of an eccentrically-positioned charge. This we do by systematic elaboration of methods borrowed from the preceding section. Immediately (which is to say: by the Law of Cosines)

$$\varphi(\boldsymbol{x}) = \tfrac{1}{4\pi}q\frac{1}{\sqrt{r^2 - 2rr\cos\vartheta + r^2}} \tag{511}$$

which—in preparation for implementation of our plan, which is to proceed by power series expansion—we will write

$$= \begin{cases} \tfrac{1}{4\pi}q\tfrac{1}{r}\cdot\dfrac{1}{\sqrt{1-2\big(\tfrac{r}{r}\big)\cos\vartheta + \big(\tfrac{r}{r}\big)^2}} & : \quad \text{adapted to the case } r < r \\ \tfrac{1}{4\pi}q\tfrac{1}{r}\cdot\dfrac{1}{\sqrt{1-2\big(\tfrac{r}{r}\big)\cos\vartheta + \big(\tfrac{r}{r}\big)^2}} & : \quad \text{adapted to the case } r > r \end{cases}$$

Thus do we acquire interest in the objects $P_n(w)$ that arise as coefficients from the series

$$\frac{1}{\sqrt{1-2wt+t^2}} = \sum_{n=0}^{\infty} P_n(w)\,t^n \tag{512.1}$$

Mathematica supplies

$$\left.\begin{aligned} P_0(w) &= 1\\ P_1(w) &= w\\ P_2(w) &= \tfrac{1}{2}(3w^2-1)\\ P_3(w) &= \tfrac{1}{2}(5w^3-3w)\\ P_4(w) &= \tfrac{1}{8}(35w^4-30w^2+3)\\ P_5(w) &= \tfrac{1}{8}(63w^5-70w^3+15w)\\ &\ \vdots \end{aligned}\right\} \tag{512.2}$$

These are precisely the **Legendre polynomials**, the properties of which were first described (1784) by A. M. Legendre (1752–1833) and are summarized in every mathematical handbook.[311] Graphs of some low-order Legendre polynomials are shown in Figure 159.

Returning with this information to (511) we have

$$\varphi(\boldsymbol{x}) = \begin{cases} \frac{1}{4\pi}q\frac{1}{r}\cdot\sum_{n=0}^{\infty}\left(\frac{r}{r}\right)^n P_n(\cos\vartheta) & \text{in the far zone}\\[2ex] \frac{1}{4\pi}q\frac{1}{r}\cdot\sum_{n=0}^{\infty}\left(\frac{r}{r}\right)^n P_n(\cos\vartheta) & \text{in the near zone} \end{cases} \tag{513}$$

in which connection it becomes pertinent to notice that (ask *Mathematica*)

$$\left.\begin{aligned} P_0(\cos\vartheta) &= 1\\ P_1(\cos\vartheta) &= \cos\vartheta\\ P_2(\cos\vartheta) &= \tfrac{1}{4}(3\cos 2\vartheta+1)\\ P_3(\cos\vartheta) &= \tfrac{1}{8}(5\cos 3\vartheta+3\cos\vartheta)\\ P_4(\cos\vartheta) &= \tfrac{1}{64}(35\cos 4\vartheta+20\cos 2\vartheta+9)\\ P_5(\cos\vartheta) &= \tfrac{1}{128}(63\cos 5\vartheta+35\cos 3\vartheta+30\cos\vartheta)\\ &\ \vdots \end{aligned}\right\} \tag{512.3}$$

Looking specifically/explicitly to the far zone we have

$$\varphi(\boldsymbol{x}) = \tfrac{1}{4\pi}\Big\{r^{-1}q + r^{-2}qrP_1(\cos\vartheta) + r^{-3}qr^2P_2(\cos\vartheta)+\cdots\Big\} \tag{514}$$

which must comprise the <u>multipole expansion—correct to all orders—of an eccentrically placed monopole</u>. How does this result compare with what (508) has to say in such a specialized situation? Setting $\rho(\boldsymbol{x}) = q\delta(\boldsymbol{x}-\boldsymbol{x})$ and working

---

[311] See, for example, W. Magnus & F. Oberhettinger, *Formulas & Theorems for the Functions of Mathematical Physics* (1949), pages 50–59; J. Spanier & K. B. Oldham, *An Atlas of Functions* (1987), Chapter 21; M. Abramowitz & Irene Stegun, *Handbook of Mathematical Functions* (1964), Chapter 22. For discussion of how the principal properties of the Legendre polynomials are established see pages 471–475 in CLASSICAL ELECTRODYNAMICS (1980).

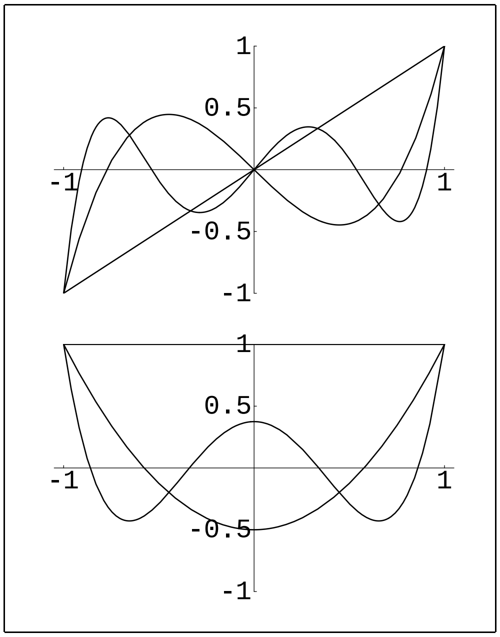


FIGURE 159: *Graphs of Legendre polynomials of low odd order (above) and low even order (below). Order can in each case be determined by counting the number of zero-crossings. The* $P_n(w)$ *are orthogonal in the sense*

$$\int_{-1}^{+1} P_m(w)P_n(w)\,dw = \tfrac{2}{2m+1}\delta_{mn}$$

and provide a natural basis within the space of functions defined on the interval $\big[-1,+1\big]$.

from (508), we find that

$$q \equiv \iiint q\delta(\boldsymbol{x}-\boldsymbol{x})\,d^3x = q$$

$$= qP_0(\cos\vartheta) \quad : \quad \text{monopole terms agree trivially}$$

$$\boldsymbol{p} \equiv \iiint \begin{pmatrix} x \\ y \\ z \end{pmatrix} q\delta(\boldsymbol{x}-\boldsymbol{x})\,d^3x$$

$$= q\begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \text{so } \boldsymbol{p}\boldsymbol{\cdot}\hat{\boldsymbol{x}} = qr\cos\vartheta \quad \text{by definition of } \vartheta$$

$$= qrP_1(\cos\vartheta) \quad : \quad \text{dipole terms agree}$$

and finally that

$$\begin{aligned}
\mathbb{Q} &\equiv \iiint \begin{pmatrix} 3x^2 - r^2 & 3xy & 3xz \\ 3yx & 3y^2 - r^2 & 3yz \\ 3zx & 3zy & 3z^2 - r^2 \end{pmatrix} q\delta(\boldsymbol{x} - \boldsymbol{x})\, d^3x \\
&= q \begin{pmatrix} 3x^2 - r^2 & 3xy & 3xz \\ 3yx & 3y^2 - r^2 & 3yz \\ 3zx & 3zy & 3z^2 - r^2 \end{pmatrix} \\
&\Downarrow \\
\tfrac{1}{2}\hat{\boldsymbol{x}}\cdot\mathbb{Q}\hat{\boldsymbol{x}} &= q\Big\{\tfrac{3}{2}(\boldsymbol{x}\cdot\hat{\boldsymbol{x}})^2 - \tfrac{1}{2}r^2\Big\} \\
&= qr^2\tfrac{1}{2}(\cos^2\vartheta - 1) \\
&= qr^2 P_2(\cos\vartheta)
\end{aligned}$$

So though (508) and (514) look quite different, they do in fact say exactly the same thing. Which is gratifying, but . . .

Equation (514) says in its complicated way what we could say quite simply if we were to reposition our coordinate system (place the origin *at* the solitary charge), so is of relatively little interest in itself. It acquires profound interest, however, when put to its intended use:

4. Representation of an arbitrary potential by superimposed spherical harmonics. The idea is to apply (514) to each constituent element $\rho(\boldsymbol{x})\, d^3x$ of our distributed charge. To implement the idea we introduce spherical coordinates in the usual way

$$\boldsymbol{x} = r\begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}, \qquad \boldsymbol{x} = r\begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}$$

where evidently θ signifies *co*latitude (North and South poles are coordinated $\theta = 0$ and $\theta = \pi$, respectively). Then

$$\cos\vartheta = \hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{x}} = \cos\theta\cos\theta + \sin\theta\sin\theta\cos(\phi - \phi)$$

and

$$d^3x = r^2 \sin\theta\, dr d\theta d\phi$$

so (514) supplies

$$\begin{aligned}
\varphi(\boldsymbol{x}) = \tfrac{1}{4\pi}r^{-1}\sum_{n=0}^{\infty}\iiint \big(\tfrac{r}{r}\big)^n P_n\big(\cos\theta\cos\theta + \sin\theta\sin\theta\cos(\phi - \phi)\big) \\
\cdot\, \rho(r, \theta, \phi) r^2 \sin\theta\, dr d\theta d\phi \qquad (515)
\end{aligned}$$

Thumbing through the mathematical handbooks, we discover the wonderful identity[312]

$$\begin{aligned}&P_n\big(\cos\theta\cos\theta+\sin\theta\sin\theta\cos(\phi-\phi)\big) \quad (516.1)\\&\quad=P_n(\cos\theta)P_n(\cos\theta)+2\sum_{m=0}^n\frac{(n-m)!}{(n+m)!}P_n^m(\cos\theta)P_n^m(\cos\theta)\cos m(\phi-\phi)\end{aligned}$$

Here

$$\begin{aligned}P_n^m(w)&\equiv(-)^m(1-w^2)^{\frac{1}{2}m}\big(\tfrac{d}{dw}\big)^m P_n(w)\quad:\quad m=0,1,2,\ldots,n\\P_n(w)&\equiv(-)^n\tfrac{1}{2^n n!}\big(\tfrac{d}{dw}\big)^n(1-w^2)^n\end{aligned}$$

defines the so-called *associated Legendre functions*, the first few of which are displayed below:[313]

$$\begin{aligned}
P_0(w)\equiv P_0^0(w)&=1 &&=1\\[1em]
P_1(w)\equiv P_1^0(w)&=w &&=\cos\theta\\
P_1^1(w)&=-\sqrt{1-w^2} &&=-\sin\theta\\[1em]
P_2(w)\equiv P_2^0(w)&=\tfrac12(3w^2-1) &&=\tfrac14(3\cos2\theta+1)\\
P_2^1(w)&=-3w\sqrt{1-w^2} &&=-\tfrac32\sin2\theta\\
P_2^2(w)&=-3(w^2-1) &&=-\tfrac32(\cos2\theta-1)\\[1em]
P_3(w)\equiv P_3^0(w)&=\tfrac12(5w^3-3w) &&=\tfrac18(5\cos3\theta+3\cos\theta)\\
P_3^1(w)&=-\tfrac32(5w^2-1)\sqrt{1-w^2} &&=-\tfrac38(5\sin3\theta+\sin\theta)\\
P_3^2(w)&=15w(1-w^2) &&=-\tfrac{15}{4}(\cos3\theta-\cos\theta)\\
P_3^3(w)&=-15(1-w^2)\sqrt{1-w^2} &&=-\tfrac{15}{4}(\sin3\theta-3\sin\theta)
\end{aligned}$$

I have written these out to demonstrate that, while $P_n^m(w)$ is a polynomial only if m is even, the associated Legendre functions are in all cases simple

---

[312] Magnus & Oberhettinger,[311] page 55; P. Morse & H. Feshbach, *Methods of Theoretical Physics* (1953), page 1274. Identities of the frequently-encountered design

$$f(x+y)=\sum_n g_n(x)g_n(y)$$

are called "addition formulæ."

[313] Use *Mathematica* to reproduce/extend the list. The commands are

```
LegendreP[n,m,w]  and  LegendreP[n,m,Cos[θ]]//TrigReduce
```

combinations of elementary functions—nothing to become nervous about. If we now write

$$\cos m(\phi - \phi) = \frac{e^{im(\phi-\phi)} + e^{-im(\phi-\phi)}}{2}$$

and accept the convention[314] that

$$P_n^m(w) \quad \text{and} \quad P_n^{-m}(w) \quad \text{are two names for the same thing}$$

then (516.1) becomes

$$\begin{aligned} P_n\big(\cos\theta\cos\theta + \sin\theta\sin\theta\cos(\phi-\phi)\big) \\ = \sum_{m=-n}^{m=+n} C_n^m \cdot P_n^m(\cos\theta)e^{-im\phi} \cdot P_n^m(\cos\theta)e^{+im\phi} \\ C_n^m \equiv \frac{(n-|m|)!}{(n+|m|)!} \end{aligned} \tag{516.2}$$

in which the (θ,ϕ)-variables and (θ,ϕ)-variables have been fully disentangled, placed in nearly identical "piles." Further simplifications become possible when one reflects upon the orthogonality properties of $e^{im\phi}$ and $P_n^m(w)$. Familiarly

$$\int_0^{2\pi} e^{-im\phi}e^{+im\phi} = 2\pi\,\delta_{mm}$$

Less familiarly—but as the handbooks inform us, and as (even in the absence of explicit proof) we are readily convinced by a little *Mathematica*-assisted experimentation—

$$\int_{-1}^{+1} P_n^m(w)P_n^m(w) = \tfrac{2}{2n+1}C_n^m\,\delta_{nn} \quad : \quad 0 \leqslant m \leqslant \text{lesser of } n \text{ and } n$$

So we construct

$$\mathcal{Y}_n^m(w,\phi) \equiv \underbrace{(-)^m}_{\text{a convention, fairly standard to the literature, and honored by } Mathematica}\sqrt{\tfrac{2n+1}{4\pi}\tfrac{(n+|m|)!}{(n-|m|)!}}\,P_n^m(w)e^{im\phi} \quad : \quad m = 0, \pm1, \pm2, \ldots, \pm n$$

which are orthonormal in the sense

$$\int_0^{2\pi}\!\!\int_{-1}^{+1} [\mathcal{Y}_n^m(w,\phi)]^*\mathcal{Y}_n^m(w,\phi)\,dwd\phi = \delta^{mm}\delta_{nn}$$

Or—more suitably for the matter at hand—

$$Y_n^m(\theta,\phi) \equiv \mathcal{Y}_n^m(\cos\theta,\phi)$$

[314] Beware! The designers of *Mathematica* adopted at this point an alternative convention.

which are precisely the celebrated **spherical harmonics**, orthonormal on the surface of the sphere

$$\int_0^{2\pi}\!\!\int_0^{\pi} [Y_n^m(\theta,\phi)]^* Y_n^m(\theta,\phi)\,\sin\theta d\theta d\phi = \delta^{mm}\delta_{nn}$$

just as the functions $E_m(\phi) \equiv \frac{1}{\sqrt{2\pi}}e^{im\phi}$ were seen above to be orthonormal on the surface of the circle. The functions $Y_n^m(\theta,\phi)$ are relatively more complicated than the functions $E_m(\phi)$ not so much because they have an extra argument as because *the surface of a sphere is a topologically more complicated place than the surface of a circle* (or—more aptly—than the surface of a torus). *Mathematica*, upon the command `SphericalHarmonicY[n,m,θ,ϕ]`, produces the following explicit list of low-order spherical harmonics:

$$Y_0^0(\theta,\phi) = \sqrt{\tfrac{1}{4\pi}}$$

$$\begin{aligned}
Y_1^{-1}(\theta,\phi) &= +\sqrt{\tfrac{3}{8\pi}}e^{-i\phi}\sin\theta\\
Y_1^{0}(\theta,\phi) &= \sqrt{\tfrac{3}{4\pi}}\cos\theta\\
Y_1^{+1}(\theta,\phi) &= -\sqrt{\tfrac{3}{8\pi}}e^{+i\phi}\sin\theta
\end{aligned}$$

$$\begin{aligned}
Y_2^{-2}(\theta,\phi) &= +\sqrt{\tfrac{15}{32\pi}}e^{-2i\phi}\sin^2\theta\\
Y_2^{-1}(\theta,\phi) &= +\sqrt{\tfrac{15}{8\pi}}e^{-i\phi}\cos\theta\sin\theta\\
Y_2^{0}(\theta,\phi) &= +\sqrt{\tfrac{5}{16\pi}}(3\cos^2\theta-1)\\
Y_2^{+1}(\theta,\phi) &= -\sqrt{\tfrac{15}{8\pi}}e^{+i\phi}\cos\theta\sin\theta\\
Y_2^{+2}(\theta,\phi) &= +\sqrt{\tfrac{15}{32\pi}}e^{+2i\phi}\sin^2\theta
\end{aligned}$$

There are $2n+1 = 1, 3, 5, \ldots$ of the things of order $n = 0, 1, 2, \ldots$

By this point (516.2) has assumed the form

$$\begin{aligned}
P_n\big(\cos\theta\cos\theta + \sin\theta\sin\theta\cos(\phi-\phi)\big)\\
= \sum_{m=-n}^{m=+n} \tfrac{4\pi}{2n+1}[Y_n^m(\theta,\phi)]^* Y_n^m(\theta,\phi)
\end{aligned}\tag{516.3}$$

which when introduced into (515) gives

$$\varphi(\boldsymbol{x}) = \tfrac{1}{4\pi}r^{-1}\sum_{n=0}^{\infty}\tfrac{4\pi}{2n+1}\sum_{m=-n}^{m=+n} Q_n^m\,\frac{Y_n^m(\theta,\phi)}{r^n}\tag{517}$$

where
$$Q_m^n \equiv \iiint [Y_n^m(\theta,\phi)]^*\rho(r,\theta,\phi)r^{n+2}\sin\theta\, dr d\theta d\phi$$

defines the **multipole moments of the charge distribution**:

$$
\begin{array}{ccccccc}
 & & & Q_0^0 & & & \\
 & & Q_1^{-1} & Q_1^0 & Q_1^{+1} & & \\
 & Q_2^{-2} & Q_2^{-1} & Q_2^0 & Q_2^{+1} & Q_2^{+2} & \\
 & & & \vdots & & & \\
Q_n^{-n} & \ldots\ldots\ldots\ldots & Q_n^{-1} & Q_n^0 & Q_n^{+1} & \ldots\ldots\ldots\ldots & Q_n^{+n}
\end{array}
$$

To remove any element of the mystery from the situation let us look to some of the illustrative specifics:

$$
\begin{aligned}
Q_0^0 &= \iiint [Y_0^0(\theta,\phi)]^* \rho(r,\theta,\phi) r^2 \sin\theta \, dr d\theta d\phi \\
&= \sqrt{\tfrac{1}{4\pi}} \iiint \rho(r,\theta,\phi) r^2 \sin\theta \, dr d\theta d\phi \\
&= \sqrt{\tfrac{1}{4\pi}}\, q
\end{aligned}
\tag{518$_0^0$}
$$

$$
\begin{aligned}
Q_1^0 &= \iiint [Y_1^0(\theta,\phi)]^* \rho(r,\theta,\phi) r^3 \sin\theta \, dr d\theta d\phi \\
&= \sqrt{\tfrac{3}{4\pi}} \iiint r\cos\theta \cdot \rho(r,\theta,\phi) r^2 \sin\theta \, dr d\theta d\phi \\
&= \sqrt{\tfrac{3}{4\pi}} \iiint z \cdot \rho(\boldsymbol{x})\, d^3x \\
&= \sqrt{\tfrac{3}{4\pi}}\, p_3
\end{aligned}
\tag{518$_1^0$}
$$

$$
\begin{aligned}
Q_1^{-1} &= \iiint [Y_1^{-1}(\theta,\phi)]^* \rho(r,\theta,\phi) r^3 \sin\theta \, dr d\theta d\phi \\
&= +\sqrt{\tfrac{3}{8\pi}} \iiint r(\cos\phi - i\sin\phi)^* \sin\theta \cdot \rho(r,\theta,\phi) r^2 \sin\theta \, dr d\theta d\phi \\
&= +\sqrt{\tfrac{3}{8\pi}} \iiint (x+iy) \cdot \rho(\boldsymbol{x})\, d^3x \\
&= +\sqrt{\tfrac{3}{8\pi}}\, (p_1 + ip_2)
\end{aligned}
\tag{518$_1^{-1}$}
$$

$$
Q_1^{+1} = -\sqrt{\tfrac{3}{8\pi}}\, (p_1 - ip_2) \tag{518$_1^{+1}$}
$$

$$
\begin{aligned}
Q_2^0 &= \iiint [Y_2^0(\theta,\phi)]^* \rho(r,\theta,\phi) r^4 \sin\theta \, dr d\theta d\phi \\
&= \sqrt{\tfrac{5}{16\pi}} \iiint (3z^2 - r^2) \cdot \rho(\boldsymbol{x})\, d^3x \\
&= \sqrt{\tfrac{5}{16\pi}}\, Q_{33}
\end{aligned}
\tag{518$_2^0$}
$$

$$
\begin{aligned}
Q_2^{-1} &= \iiint [Y_2^{-1}(\theta,\phi)]^* \rho(r,\theta,\phi)\, r^4 \sin\theta\, dr d\theta d\phi \\
&= +\sqrt{\tfrac{15}{8\pi}} \iiint r^2(\cos\phi + i\sin\phi)\cos\theta\sin\theta\cdot\rho(r,\theta,\phi) r^2\sin\theta\, dr d\theta d\phi \\
&= +\sqrt{\tfrac{15}{8\pi}} \iiint (x+iy)z\cdot\rho(\boldsymbol{x})\, d^3x \\
&= +\sqrt{\tfrac{15}{8\pi}}\tfrac{1}{3}(Q_{13}+iQ_{23}) && (518)_2^{-1} \\
Q_2^{+1} &= -\sqrt{\tfrac{15}{8\pi}}\tfrac{1}{3}(Q_{13}-iQ_{23}) && (518)_2^{+1} \\
Q_2^{-2} &= \iiint [Y_2^{-2}(\theta,\phi)]^* \rho(r,\theta,\phi)\, r^4 \sin\theta\, dr d\theta d\phi \\
&= +\sqrt{\tfrac{15}{32\pi}} \iiint r^2 \underbrace{(\cos 2\phi + i\sin 2\phi)}_{=\cos^2\phi - \sin^2\phi + 2i\cos\phi\sin\phi}\sin^2\theta\cdot\rho(r,\theta,\phi) r^2\sin\theta\, dr d\theta d\phi \\
&= +\sqrt{\tfrac{15}{32\pi}} \iiint (x^2-y^2+2ixy)\cdot\rho(\boldsymbol{x})\, d^3x \\
&= +\sqrt{\tfrac{15}{32\pi}}\tfrac{1}{3}(Q_{11}-Q_{22}+2iQ_{12}) && (518)_2^{-2} \\
Q_2^{+2} &= +\sqrt{\tfrac{15}{32\pi}}\tfrac{1}{3}(Q_{11}-Q_{22}-2iQ_{12}) && (518)_2^{+2}
\end{aligned}
$$

Here the notations p_a and Q_{ab} have been taken from (508) on page 424. The point is that same physical information is folded (if in a different way) into the designs of Q_1^m, Q_2^m, ... as was folded into the designs of $\boldsymbol{p}$, $\mathbb{Q}$, ...: equations (517) and (508) are saying the same thing, but in different ways.

Were we to pursue the mathematical side of this subject we would want to establish that & how the spherical harmonics $Y_n^m(\theta,\phi)$ spring spontaneously into being when one undertakes to

$$\text{solve } \nabla^2\varphi = 0 \text{ in spherical coordinates by separation of variables}$$

A little *Mathematica*-assisted experimentation[315] may serve to convince the reader—even in the absence of the formal demonstration—that

$$\nabla^2\big\{r^p\, Y_n^m(\theta,\phi)\big\} = 0 \quad \text{if and only if } p = n \text{ or } p = -(n+1)$$

---

[315] Enter the commands

```
<<Calculus`VectorAnalysis`
```

and

```
SetCoordinates[Spherical[r,θ,φ]]
```

and then test

```
Laplacian[r^p SphericalHarmonicY[n,m,θ,φ]]
```

with various values of m, n and p.

Solutions of the first type blow up as $r \uparrow \infty$: at (517) we find $\varphi(\boldsymbol{x})$ described as a linear combination of solutions of the second type. Looking to the mathematics of the situation from a somewhat different angle . . .

$$\varphi(r,\theta,\varphi) = \sum_{m,n} A_{mn}\big\{\big(\tfrac{r}{a}\big)^n\, Y_n^m(\theta,\phi)\big\}$$

describes a solution of Laplace's equation,[316] and so also does

$$\psi(r,\theta,\varphi) = \tfrac{a}{r}\sum_{m,n} A_{mn}\big\{\big(\tfrac{a}{r}\big)^n\, Y_n^m(\theta,\phi)\big\}$$

To say the same thing another way: if $f(x,y,z)$ is a solution of Laplace's equation $\nabla^2 f = 0$ then so also is

$$F(x,y,z) \equiv \tfrac{a}{r} f\big(\tfrac{a^2}{r^2}x, \tfrac{a^2}{r^2}y, \tfrac{a^2}{r^2}z\big)$$

Transformations of the form

$$\boldsymbol{x} \xrightarrow[\text{inversion}]{} \boldsymbol{x} = \tfrac{a^2}{r^2}\boldsymbol{x}$$

are called "inversions in the sphere of radius a" by geometers (they send interior points to exterior points and *visa versa*, subject to the rule $rr = a^2$), and are self-inversive in the sense

$$\boldsymbol{x} \xrightarrow[\text{inversion}]{} \tfrac{a^2}{r^2}\boldsymbol{x} = \tfrac{r^2}{a^2}\boldsymbol{x} = \boldsymbol{x}$$

Transformations of the form

$$f(\boldsymbol{x}) \xrightarrow[\text{Kelvin inversion}]{} f(\boldsymbol{x}) \equiv \tfrac{a}{r} f\big(\tfrac{a^2}{r^2}\boldsymbol{x}\big) \tag{519}$$

acquire their name from the fact that it was William Thompson (Lord Kelvin) who first noticed (1847) that they send "harmonic functions" (solutions of Laplace's equation) into harmonic functions: they are readily seen to be self-inversive in the sense that

$$(\text{Kelvin inversion})^2 = \text{identity transformation}$$

Rotation of the charge distribution (equivalently: counter rotation of the Cartesian frame) would clearly result in an altered set of coefficients Q_n^m that refer to an altered set of spherical harmonics:

$$\begin{aligned}\varphi(\boldsymbol{x}) &= \tfrac{1}{4\pi} r^{-1} \sum_{n=0}^{\infty} \tfrac{4\pi}{2n+1} \sum_{m=-n}^{m=+n} Q_n^m \frac{Y_n^m(\theta,\phi)}{r^n}\\ &\Big\downarrow \text{rotation}\\ &= \tfrac{1}{4\pi} r^{-1} \sum_{n=0}^{\infty} \tfrac{4\pi}{2n+1} \sum_{m=-n}^{m=+n} Q_n^m \frac{Y_n^m(\theta,\phi)}{r^n}\end{aligned}$$

[316] Here and below: a is a constant "length" of arbitrary value, introduced for a dimensional reason.

Were we to pursue the theory of spherical harmonics we would certainly want to explore the details of the now-fairly-evident fact that the harmonics of given order n are rotationally induced to fold among themselves

$$\begin{pmatrix} Y_n^{+n}(\theta,\phi) \\ \vdots \\ Y_n^0(\theta,\phi) \\ \vdots \\ Y_n^{-n}(\theta,\phi) \end{pmatrix} = \begin{pmatrix} \\ \\ \quad (2n+1)\times(2n+1) \text{ matrix} \quad \\ \\ \\ \end{pmatrix} \begin{pmatrix} Y_n^{+n}(\theta,\phi) \\ \vdots \\ Y_n^0(\theta,\phi) \\ \vdots \\ Y_n^{-n}(\theta,\phi) \end{pmatrix}$$

in a why that provides a (2n+1)-dimensional representation of the 3-dimensional rotation group $O(3)$. When those details are approached algebraically (instead off function-theoretically) it is found to make sense to speak also of cases

$$n = \tfrac{1}{2}, \tfrac{3}{2}, \tfrac{5}{2}, \ldots$$

that give rise to even-dimensional matrix representations of $O(3)$, and that those have indispensible applications to the quantum theory of fractional spin. While electrostatics served historically to inspire the initial development of the theory of spherical harmonics, and does exploit some of the more superficial elements of that theory, it is the *quantum theory of angular momentum* (equivalently: the representation theory of $O(3)$) that first motivated people to explore (in order to exploit) the riches hidden in the deeper nooks and crannies of the theory of spherical harmonics. And it is because the theory is most naturally developed in connection with its quantum mechanical applications[317] that I am content not to pursue it further here.[318]

5. A geophysical application. Though initially formulated in the language of electrostatics, our results pertain also—obviously and quite usefully—to gravitostatics . . . for reasons having to do with the structural similarity of the statements

$$\frac{e}{4\pi r} = \text{electrostatic potential of a point charge } e$$
$$-\frac{GM}{r} = \text{gravitostatic potential of a point mass } M$$

Evidently the gravitational potential exterior to a sphere[319] containing a blob $\rho(\boldsymbol{x})$ of matter—the earth is the "blob" of greatest interest to geophysicists—can

[317] See, for example, David Griffiths, *Introduction to Quantum Mechanics* (1995), Chapter 4 or J. Powell & B. Crasemann, *Quantum Mechanics* (1961), Chapter 7.

[318] In 1980 I had not so much self-control: the missing details are sketched on pages 486–510 of CLASSICAL ELECTRODYNAMICS.

[319] A *mental* sphere, of radius a, commonly identified with the maximal radius of the geosphere ($\sim 6.378\times 10^3$ km).

be described

$$\begin{aligned}
V(\boldsymbol{x}) &= -G\iiint \rho(\boldsymbol{x})\,\frac{1}{|\boldsymbol{x}-\boldsymbol{x}|}\,d^3x \\
&= -Gr^{-1}\sum_{n=0}^{\infty}\tfrac{4\pi}{2n+1}\sum_{m=-n}^{m=+n} Q_n^m\,\frac{Y_n^m(\theta,\phi)}{r^n} \\
&\qquad\qquad Q_m^n \equiv \iiint [Y_n^m(\theta,\phi)]^*\rho(r,\theta,\phi)r^{n+2}\sin\theta\,dr d\theta d\phi \\
&= -GM\tfrac{1}{r}\Big\{1+\tfrac{1}{r}\boldsymbol{P}\cdot\hat{\boldsymbol{x}}+\tfrac{1}{r^2}\tfrac{1}{2}\hat{\boldsymbol{x}}\cdot\mathbb{G}\hat{\boldsymbol{x}}+\cdots\Big\}
\end{aligned}$$

where

$$M \equiv \iiint \rho(\boldsymbol{x})\,d^3x = \text{monopole moment} = \textbf{total mass}$$

$$\boldsymbol{P} \equiv \tfrac{1}{M}\iiint \rho(\boldsymbol{x})\,d^3x = \frac{\text{dipole moment vector}}{M}$$

$$= \textbf{center of mass}\ \text{coordinates}$$

$$\mathbb{G} \equiv \tfrac{1}{M}\iiint \|3x^ix^j - r^2\delta^{ij}\|\rho(\boldsymbol{x})\,d^3x = \frac{\text{quadrupole moment matrix}}{M}$$

Note that *<u>the dipole term drops away</u> if one places the origin at the center of mass.*[320] Dominant interest shifts therefore to the quadrupole term, which "MacCullagh's formula"

$$V(\boldsymbol{x}) = -GM\tfrac{1}{r}\Big\{1-\frac{A-C}{2Mr^2}(3\sin^2\underset{\uparrow\text{signifies latitude}}{\psi}-1)+\cdots\Big\}$$

serves to relate to the geometrical parameters (A and C) that describe the idealized oblate sphereoidal figure of the gravitating body (see again Figure 154). Higher moments provide information about

- irregularities in the figure of the body
- inhomogeneities of the mass distribution.

Notice that (see again the formula that serves at the top of the page to define the coefficients Q_n^m) the higher moments depend most strongly upon details near the *surface* of the body, and are of quantitative significance only in the near zone: far away the body "looks like a monopole":

$$\begin{aligned}&\downarrow\\ &= -GM\tfrac{1}{r} \quad : \quad r \gg a\end{aligned}$$

For the earth the Q_n^m have been measured through at least $n = 8$, and in the post-Sputnik era satellites have been used to fill in an "island" of higher

[320] That would be a natural thing to do, but a *conventional* thing to do (something one might elect *not* to do) . . . and should not be confused with the physical fact that—because Nature provides no "negative mass"—*gravitational dipoles do not exist.*

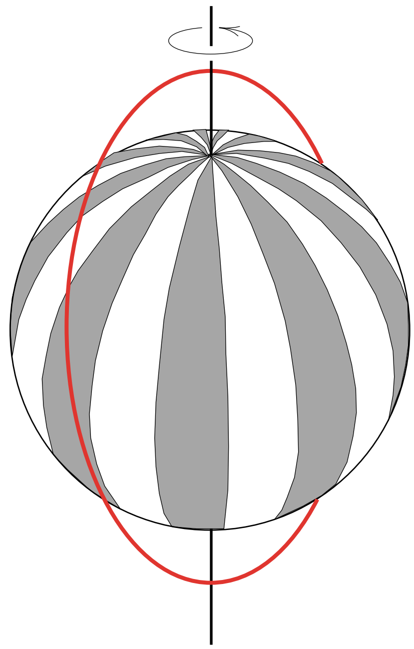


FIGURE 160: *Polar orbit of a satellite in polar orbit. Resolving the spherical harmonics into their real/imaginary parts*

$$Y_n^{\pm m}(\theta,\phi) = C_n^m(\theta,\phi) \pm iS_n^m(\theta,\phi)$$

we observe that $S_n^0(\theta,\phi)$ *and* $C_n^0(\theta,\phi)$ *are* ϕ*-independent: they vanish on circles parallel to the equator, thus partitioning the surface of the sphere into "zones," so are called "zonal harmonics." At the other extreme, the nodes of*

$$C_n^n(\theta,\phi) \sim \cos n\phi \sin^n\theta \quad \textit{and} \quad S_n^n(\theta,\phi) \sim \sin n\phi \sin^n\theta$$

partition the sphere into sectors (bounded by great arcs of constant longitude); such functions are called "sectoral harmonics," while spherical harmonics with $0<m<n$ *are called "tesseral harmonics." Some sectors have been painted on the earth, and rotate with the earth (because they are taken here to refer to a property* of the earth*).*

(m,n)-values—this by the pretty method that I now sketch. The period T of a satellite in circular orbit can, in leading approximation, be described

$$T = 2\pi\sqrt{\frac{a^3}{GM}}\big(\tfrac{r}{a}\big)^{\frac{3}{2}}$$

which in the case of the earth becomes

$$= 84.5\big(\tfrac{r}{a}\big)^{\frac{3}{2}} \text{ minutes}$$

The satellite will be in *resonance* with the sectoral harmonics Y_n^n of the earth's gravitational field if $T = T_n$, where T_n is the time it takes for the rotating earth to replace one of the sectors of Y_n^n by the next. The sidereal day is 1436.07 minutes long, so

$$T_n = \frac{1436.07 \text{ minutes}}{n} = \begin{cases} 89.75 \text{ minutes} & : \quad n = 16 \\ 95.74 \text{ minutes} & : \quad n = 15 \\ 102.58 \text{ minutes} & : \quad n = 14 \\ 110.47 \text{ minutes} & : \quad n = 13 \\ 119.67 \text{ minutes} & : \quad n = 12 \end{cases}$$

and to achieve synchrony in those cases (solve $84.5x^{\frac{3}{2}} = T_n$ for x) we must set

$$\text{orbital radius} = \begin{cases} 1.0410a & : \quad \text{resonance with } Y_{16}^{16} \text{ mode} \\ 1.0868a & : \quad \text{resonance with } Y_{15}^{15} \text{ mode} \\ 1.1380a & : \quad \text{resonance with } Y_{14}^{14} \text{ mode} \\ 1.1956a & : \quad \text{resonance with } Y_{13}^{13} \text{ mode} \\ 1.2611a & : \quad \text{resonance with } Y_{12}^{12} \text{ mode} \end{cases}$$

If $n \gtrsim 16$ the satellite burns up in the atmosphere (or its orbit becomes subterranean!), while if $n \lesssim 12$ then r becomes so large that the $(1/r^n)$-factor makes the effects of resonance unobservably small. The case $n = 15$ seems to be nearly optimal, and indeed: scientists active in the field[321] have been able by this means to estimate the values of Q_{15}^{15}, Q_{17}^{15}, Q_{19}^{15} and Q_{21}^{15}. Since high moments probe progressively more *superficial* properties of $\rho(\boldsymbol{x})$, one might hope from such orbital data to extract information about the earth's *crust and crust-mantle interface*. The technique extends in principle to planetary bodies other than the earth. And microphysical analogs do come to mind: an atom with nuclear charge Ze has orbital radii given typically by (see again page 392)

$$R = \frac{\hbar^2}{mZe^2}$$

which *gets smaller when m is increased*. One therefore expects that *the properties of μ-mesonic atoms might provide information about the surface properties of complex nuclei*.

6. Harmonic polynomials & Maxwell's theory of poles. While the theory of spherical harmonics has much to do with the representation of rotations in 3-space, it has—contrary to the impression conveyed by some of the preceding material—only incidentally to do with spherical coordinates. Important aspects of the theory are, in fact, brought most simply/naturally into view by the adoption of a Cartesian perspective . . . as I undertake now to demonstrate:

---

[321] See R. D. Eberst, "Earth satellites and the gravitational potential" and D. G. King-Hele & H. Heller, "Equations for the 15th-order harmonics in the geopotential," Nature Physical Science **235**, 130 (1972). Also A. E. Roy, *Orbital Motion* §10.4 (1978) and H. F. R. Schöyer & K. F. Walker, *Rocket Propulsion and Space Flight Dynamics* §18.6 (1979).

Introduce the (rotationally invariant!) monomial $T(\boldsymbol{x}) \equiv \boldsymbol{a}\cdot\boldsymbol{x}$ and notice that, by quick calculation,

$$\nabla^2 T^n = n(n-1)T^{n-2}\,\boldsymbol{a}\cdot\boldsymbol{a}$$

Dismissing as trivial the cases $n = 0$ and $n = 1$, we conclude that <u>the n^{th} powers of $T(\boldsymbol{x})$ will be harmonic iff $\boldsymbol{a}$ is null</u>. But $\boldsymbol{a}\cdot\boldsymbol{a} = 0$ entails that $\boldsymbol{a}$ be complex: $\boldsymbol{a} = \boldsymbol{b} + i\boldsymbol{c}$ with $b^2 - c^2 = 0$ and $\boldsymbol{b}\cdot\boldsymbol{c} = 0$. If $\boldsymbol{a}\cdot\boldsymbol{a} = 0$ is formulated

$$a_3 = \sqrt{-(a_1^2 + a_2^2)} = i\sqrt{(a_1 + ia_2)(a_1 - ia_2)}$$

then it becomes fairly natural to introduce complex parameters

$$\begin{aligned} u &\equiv \sqrt{a_1 + ia_2} \\ v &\equiv \sqrt{a_1 - ia_2} \end{aligned}$$

in terms of which we can write

$$\left.\begin{aligned} a_1 &= \tfrac{1}{2}(u^2 + v^2) \\ a_2 &= \tfrac{1}{2i}(u^2 - v^2) \\ a_3 &= iuv \end{aligned}\right\} \tag{520}$$

which provide a (u, v)-parameterized description of the set of all null 3-vectors $\boldsymbol{a}$. In this notation

$$\begin{aligned} T^n(\boldsymbol{x}) &= \tfrac{1}{2^n}\big[(u^2 + v^2)x + \tfrac{1}{i}(u^2 - v^2)y + 2iuvz\big]^n \\ &= \tfrac{1}{2^n}\big[u^2(x - iy) + 2iuvz + v^2(x + iy)\big]^n \\ &= \begin{cases} \text{polynomial of degree } \ \ n \text{ in variables } (x, y, z) \\ \text{polynomial of degree } 2n \text{ in parameters } (u, v) \end{cases} \end{aligned}$$

To emphasize the latter point of view we write

$$= \tfrac{1}{2^n}\sum_{m=-n}^{m=+n} u^{n-m}v^{n+m}H_n^m(\boldsymbol{x})$$

This, since harmonic for *all* values of u and v, entails that the polynomials $H_n^m(\boldsymbol{x})$ are *individually* harmonic:

$$\nabla^2 H_n^m = 0$$

Arguing from

$$\begin{aligned}
T\cdot T^n &= \tfrac{1}{2^{n+1}}\big[u^2(x-iy)+2iuvz+v^2(x+iy)\big]\sum_m u^{n-m}v^{n+m}H_n^m\\
&= \tfrac{1}{2^{n+1}}\sum_m\Big\{\quad u^{(n+1)-(m-1)}v^{(n+1)+(m-1)}(x-iy)H_n^m\\
&\qquad\qquad + u^{(n+1)-m}\qquad v^{(n+1)+m}\qquad (2iz)H_n^m\\
&\qquad\qquad + u^{(n+1)-(m+1)}v^{(n+1)+(m+1)}(x+iy)H_m^m\Big\}\\
&= T^{(n+1)}\\
&= \tfrac{1}{2^{n+1}}\sum_m u^{(n+1)-m}v^{(n+1)+m}H_{n+1}^m
\end{aligned}$$

we obtain a relation

$$H_{n+1}^m = (x-iy)H_n^{m+1}+2izH_n^m+(x+iy)H_n^{m-1}$$

from which—sprouting from the "seed"

$$H_0^m(\boldsymbol{x})\equiv\begin{cases}1 & : \quad m=0\\ 0 & : \quad m=\pm1,\pm2,\ldots\end{cases}$$

—the **harmonic polynomials** $H_n^m(\boldsymbol{x})$ can be computed recursively: thus

$$\begin{aligned}
H_0^0\ &= 1\\
\\
H_1^{-1} &= x-iy\\
H_1^0\ &= 2iz\\
H_1^{+1} &= x+iy\\
\\
H_2^{-2} &= (x-iy)^2\\
H_2^{-1} &= 4i(x-iy)z\\
H_2^0\ &= 2x^2+2y^2-4z^2 = 2(r^2-3z^2)\\
H_2^{+1} &= 4i(x+iy)z\\
H_2^{+2} &= (x+iy)^2\\
&\ \vdots
\end{aligned}$$

The harmonic polynomials are <u>regular at the origin but blow up at ∞</u>. Kelvin inversion (519) permits us, however, to construct from them a population of (non-polynomial) functions

$$J_n^m(\boldsymbol{x})\equiv\tfrac{1}{r}H_n^m(\tfrac{1}{r^2}\boldsymbol{x})$$

which are assuredly *also harmonic* and, though singular at the origin, are regular at ∞. Reading from the preceding list are led thus to the Kelvin transform of that list:

$$
\begin{aligned}
J_0^0 &= r^{-1} && = +\tfrac{1}{r} \\[6pt]
J_1^{-1} &= r^{-3}\cdot(x-iy) && = -1(\partial_x - i\partial_y)\tfrac{1}{r} \\
J_1^{0} &= r^{-3}\cdot 2iz && = -2(i\partial_z)\tfrac{1}{r} \\
J_1^{+1} &= r^{-3}\cdot(x+iy) && = -1(\partial_x + i\partial_y)\tfrac{1}{r} \\[6pt]
J_2^{-2} &= r^{-5}\cdot(x-iy)^2 && = +\tfrac{1}{3}\cdot 1(\partial_x - i\partial_y)^2\tfrac{1}{r} \\
J_2^{-1} &= r^{-5}\cdot 4(x-iy)iz && = +\tfrac{1}{3}\cdot 4(\partial_x - i\partial_y)(i\partial_z)\tfrac{1}{r} \\
J_2^{0} &= r^{-5}\cdot 2(r^2-3z^2) && = +\tfrac{1}{3}\cdot 6(i\partial_z)^2\tfrac{1}{r} \\
J_2^{+1} &= r^{-5}\cdot 4(x+iy)iz && = +\tfrac{1}{3}\cdot 4(\partial_x + i\partial_y)(i\partial_z)\tfrac{1}{r} \\
J_2^{+2} &= r^{-5}\cdot(x+iy)^2 && = +\tfrac{1}{3}\cdot 1(\partial_x + i\partial_y)^2\tfrac{1}{r} \\
&\ \vdots && \ \vdots
\end{aligned}
$$

That the harmonic functions $J_n^m(\boldsymbol{x})$ can be described by the highly patterned formulæ on the right was discovered by Maxwell, who in the general case would have us write

$$J_n^{\pm m} = (-)^n \frac{1}{1\cdot 3\cdot 5\cdots(2n-1)}\binom{2n}{n-m}(\partial_x \pm i\partial_y)^m(i\partial_z)^{n-m}\,\tfrac{1}{r}$$

where now $m = 0, 1, 2, \ldots, n$.

We are by now not surprised to discover that if we *at this point* use

$$x \pm iy = r\sin\theta\cdot e^{\pm i\phi} \quad \text{and} \quad z = r\cos\theta$$

to pass from Cartesian to spherical coordinates, then the functions J_n^m turn out to differ only numerical factors from the functions $r^{-(n+1)}Y_n^m(\theta,\phi)$. The detailed result can be expressed in several ways:

$$
\begin{aligned}
Y_n^{\pm m}(\theta,\phi) &= (-)^n(i)^{n+m}\tfrac{1}{2^n n!}\sqrt{\tfrac{2n+1}{4\pi}(n-m)!(n+m)!}\cdot r^{n+1}J_n^{\pm m}(\boldsymbol{x}) \\
\big(\tfrac{1}{r}\big)^{n+1}Y_n^{\pm m}(\theta,\phi) &= \underbrace{(-)^n\sqrt{\tfrac{2n+1}{4\pi}\tfrac{1}{(n-m)!(n+m)!}}(\partial_x \pm i\partial_y)^m(\partial_z)^{n-m}}\ \tfrac{1}{r}
\end{aligned}
$$

From the latter we conclude that

$$\equiv \mathcal{D}_n^{\pm m}$$

is a *differential operator natural to the theory of spherical harmonics*.

Which brings us back again to very nearly our point of departure. We established at (10.2) on page 12 that the function $\frac{1}{r} = (x^2 + y^2 + z^2)^{-\frac{1}{2}}$ is harmonic except at the origin, where it blows up, but in a very interesting way:

$$\nabla^2 \tfrac{1}{r} = -4\pi\delta(\boldsymbol{x})$$

Application of $\mathcal{D}_n^{\pm m}$ gives

$$\downarrow$$

$$\nabla^2 \big(\tfrac{1}{r}\big)^{n+1} Y_n^{\pm m}(\theta,\phi) = -4\pi\mathcal{D}_n^{\pm m}\delta(\boldsymbol{x}) \tag{521}$$

which shows that a similar remark pertains to the functions $Y_n^m(\theta,\phi)/r^{n+1}$, except that these possess *singularities of higher order*, the latter being described by fancy *derivatives* of δ-functions. When, as at (517), we display $\varphi(\boldsymbol{x})$ as a weighted superposition of the functions that appear on the left, we are in effect claiming that $\rho(\boldsymbol{x})$ is equivalent to an identically weighted superposition of the singular functions ("distributions") that appear on the right side of (521):

$$\varphi(\boldsymbol{x}) = \sum_{n=0}^{\infty} \tfrac{1}{2n+1} \sum_{m=-n}^{m=+n} Q_n^m \, \frac{Y_n^m(\theta,\phi)}{r^{n+1}}$$

strength of $\mathcal{D}_n^{\pm m}\delta(\boldsymbol{x})$ singularity

number of n^{th}-order singularities

And we remarked already on page 426 the sense in which structured singularities can be interpreted to refer to constellations of "poles." We have arrived thus at the essence of Maxwell's "theory of poles."

It is hard to let go of this beautiful subject. I allow myself the luxury of one parting shot: It is an immediate implication of (520) that

$$\boldsymbol{a}^*\!\cdot \boldsymbol{a} = \tfrac{1}{2}(u^*u + v^*v)$$

The expression on the right is invariant under linear transformations

$$\begin{pmatrix} u \\ v \end{pmatrix} \longrightarrow \begin{pmatrix} u \\ v \end{pmatrix} = \mathbb{U}\begin{pmatrix} u \\ v \end{pmatrix}$$

provided $\mathbb{U}$ is *unitary* (inverse = conjugate transpose). Such transformations, by (520), induce linear transformations

$$\boldsymbol{a} \longrightarrow \boldsymbol{a} = \mathbb{R}\boldsymbol{a}$$

which, since norm-preserving, must describe 3-dimensional rotations. From this germ of an idea one gains direct access to the rich subject matter to which I allude at the end of §4.[322]

[322] Some of the details are developed in my "Algebraic theory of spherical harmonics" (Seminar Notes 1996). An excellent source is A. Erdélyi *et al*, *Higher Transcendental Functions* (1953), Volume 2, Chapter 11.

The material described above—fruit of the genius mainly of Maxwell and his friends, and of the generation that preceded them—takes Laplace's equation

$$\nabla^2\varphi = 0$$

as its point of departure, but analogous methods are important in a variety of other contexts. Look, for example, to the heat 1-dimensional equation

$$(\partial_x^2 - \partial_t)\varphi(x,t) = 0$$

It is clear that $e^{xz + tz^2}$ describes a z-parameterized family of solutions. Taylor expansion in z

$$\begin{aligned} e^{xz + tz^2} &= 1 + xz + \tfrac{1}{2}(x^2 + 2t)z^2 \\ &\qquad\qquad + \tfrac{1}{6}(x^3 + 6xt)z^3 \\ &\qquad\qquad + \tfrac{1}{24}(x^4 + 12x^2t + 12t^2)z^4 + \cdots \\ &\equiv \sum_{n=0}^{\infty} v_n(x,t)\tfrac{1}{n!}z^n \end{aligned}$$

gives rise to a population of "heat polynomials," analogous to the harmonic polynomials encountered on page 444.[323] And corresponding to the Kelvin transformation (519) one has the (nearly inversive) **Appell transformation** (1892)

$$\varphi(x,t) \xrightarrow[\text{Appell transformation}]{} \psi(x,t) \equiv \frac{e^{-x^2/4t}}{\sqrt{4\pi t}} \cdot \varphi(\tfrac{x}{t}, -\tfrac{1}{t})$$

where the exponential factor is itself a solution—the so-called "fundamental solution"—of the heat equation. We have seen that the Kelvin transformation contributes importantly to the theory of harmonic functions. Just so the Appell transformation: I have shown elsewhere that it is an object central to the theory of the conformal group, and that in a quantum mechanical application it serves as the bridge that links the standard formalism to the Feynman formalism.[324]

[323] See D. V. Widder, *The Heat Equation* (1975), pages 8–14.

[324] "Appell, Galilean & Conformal Transformations in Classical/Quantum Free Particle Dynamics" (RESEARCH NOTES 1976).